# An approximate dual subgradient algorithm for distributed non-convex constrained optimization

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***Abstract*—We consider a multi-agent optimization problem where agents aim to cooperatively minimize a sum of local objective functions subject to a global inequality constraint and a global state constraint set. In contrast to existing papers, we do not require that the objective, constraint functions, and state constraint sets are convex. We propose a distributed approximate dual subgradient algorithm to enable agents to asymptotically converge to a pair of approximate primal-dual solutions over dynamically changing network topologies. Convergence can be guaranteed provided that the Slater's condition and strong duality property are satisfied.**

## I. Introduction

Recent advances in computation, communication, sensing and actuation have stimulated an intensive research in networked multi-agent systems. In the systems and control community, this has been translated into how to solve global control problems, expressed by global objective functions, by means of local agent actions. More specifically, problems considered include multi-agent consensus or agreement [4], [10], [12], [16], [21], [22], coverage control [5], [6], formation control [7], [25] and sensor fusion [28].

In the optimization community, a problem of focus is to minimize a sum of local objective functions by a group of agents, where each function depends on a common global decision vector and is only known to a specific agent. This problem is motivated by others in distributed estimation [20] [27], distributed source localization [23], and network utility maximization [13]. More recently, consensus techniques have been proposed to address the issues of switching topologies in networks and non-separability in objective functions; see for instance [11], [18], [19], [24], [29]. More specifically, the paper [18] presents the first analysis of an algorithm that combines average consensus schemes with subgradient methods. Using projection in the algorithm of [18], the authors in [19] further solve a more general setup that takes local state constraint sets into account. Further, in [29] we develop two distributed primal-dual subgradient algorithms, which are based on saddle-point theorems, to analyze a more general situation that incorporates global inequality and equality constraints. The aforementioned algorithms are extensions of classic (primal or primal-dual) subgradient methods which generalize gradient-based methods to minimize nonsmooth functions. This requires the optimization problems under consideration to be convex in order to determine a global optimum.

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The focus of the current paper is to relax the convexity assumption in [29]. The challenges induced by the presence of non-convexity will be circumvented by the integration of Lagrangian dualization and subgradient schemes. These two techniques have been popular and efficient approaches to solve large-scale, structured convex optimization problems, e.g., [2], [3]. However, subgradient methods do not automatically generate primal solutions for nonsmooth convex optimization problems, and numerous approaches have been designed to construct primal solutions; e.g., by removing the nonsmoothness [26], by employing ascent approaches [14], and the generation of ergodic sequences [15], [17].

*Statement of Contributions.* Here, we investigate a multi-agent optimization problem where agents are trying to minimize a sum of local objective functions subject to a global inequality constraint and a global state constraint set. The objective and constraint functions as well as the state-constraint set could be non-convex. A distributed approximate dual subgradient algorithm is introduced to find a pair of approximate primal-dual solutions. Specifically, the update rule for dual estimates combines an approximate dual subgradient scheme with average consensus algorithms. To obtain primal solutions from dual estimates, we propose a novel recovery scheme: primal estimates are not updated if the variations induced by dual estimates are smaller than some predetermined threshold; otherwise, primal estimates are set to some solutions in dual optimal solution sets. This algorithm is shown to asymptotically converge to a pair of approximate primal-dual solutions over a class of switching network topologies under the assumptions of the Slater's condition and the strong duality property.

## II. Problem formulation and preliminaries

Consider a networked multi-agent system where agents are labeled by $i \in V := \{1, \dots, N\}$. The multi-agent system operates in a synchronous way at time instants $k \in \mathbb{N} \cup \{0\}$, and its topology will be represented by a directed weighted graph $\mathcal{G}(k) = (V, E(k), A(k))$, for $k \geq 0$. Here, $A(k) := [a^i_j(k)] \in \mathbb{R}^{N \times N}$ is the adjacency matrix, where the scalar $a^i_j(k) \geq 0$ is the weight assigned to the edge $(j, i)$, and $E(k) \subseteq V \times V \setminus \operatorname{diag}(V)$ is the set of edges with non-zero weights. The set of in-neighbors of agent $i$ at time $k$ is denoted by $\mathcal{N}_i(k) = \{j \in V \mid (j, i) \in E(k) \text{ and } j \neq i\}$. Similarly, we define the set of out-neighbors of agent $i$ at time $k$ as $\mathcal{N}^{\text{out}}_i(k) = \{j \in V \mid (i, j) \in E(k) \text{ and } j \neq i\}$. We here make the following assumptions on network communication graphs:

**Assumption 2.1 (Non-degeneracy):** There exists a constant $\alpha > 0$ such that $a^i_i(k) \geq \alpha$, and $a^i_j(k)$, for $i \neq j$,

satisfies $a_j^i(k) \in \{0\} \cup [\alpha,\ 1]$, for all $k \geq 0$.

**Assumption 2.2 (Balanced Communication):** [1]It holds that $\sum_{j\in V} a_j^i(k) = 1$ for all $i \in V$ and $k \geq 0$, and $\sum_{i\in V} a_j^i(k) = 1$ for all $j \in V$ and $k \geq 0$.

**Assumption 2.3 (Periodical Strong Connectivity):** There is a positive integer $B$ such that, for all $k_0 \geq 0$, the directed graph $(V, \bigcup_{k=0}^{B-1} E(k_0+k))$ is strongly connected.

The above network model is standard in the analysis of average consensus algorithms; e.g., see [21], [22], and distributed optimization in [19], [29]. Recently, an algorithm is given in [8] which allows agents to construct a balanced graph out of a non-balanced one under certain assumptions.

The objective of the agents is to cooperatively solve the following primal problem $(P)$:

$$\begin{aligned} &\min_{z\in\mathbb{R}^n} \sum_{i\in V} f_i(z), \\ &\text{s.t.} \quad g(z) \leq 0, \quad z \in X, \end{aligned} \tag{1}$$

where $z \in \mathbb{R}^n$ is the global decision vector. The function $f_i : \mathbb{R}^n \to \mathbb{R}$ is only known to agent $i$, continuous, and referred to as the objective function of agent $i$. The set $X \subseteq \mathbb{R}^n$, the state constraint set, is compact. The function $g : \mathbb{R}^n \to \mathbb{R}^m$ are continuous, and the inequality $g(z) \leq 0$ is understood component-wise; i.e., $g_\ell(z) \leq 0$, for all $\ell \in \{1, \ldots, m\}$, and represents a global inequality constraint. We will denote $f(z) := \sum_{i\in V} f_i(z)$ and $Y := \{z \in \mathbb{R}^n \mid g(z) \leq 0\}$. We will assume that the set of feasible points is non-empty; i.e., $X \cap Y \neq \emptyset$. Since $X$ is compact and $Y$ is closed, then we can deduce that $X \cap Y$ is compact. The continuity of $f$ follows from that of $f_i$. In this way, the optimal value $p^*$ of the problem $(P)$ is finite and $X^*$, the set of primal optimal points, is non-empty. Throughout this paper, we suppose the following Slater's condition holds:

**Assumption 2.4 (Slater's Condition):** There exists a vector $\bar{z} \in X$ such that $g(\bar{z}) < 0$. Such $\bar{z}$ is referred to as a Slater vector of the problem $(P)$.

**Remark 2.1:** All the agents can agree upon a common Slater vector $\bar{z}$ through a maximum-consensus scheme. This can be easily implemented as part of an initialization step, and thus the assumption that the Slater vector is known to all agents does not limit the applicability of our algorithm. Specifically, the maximum-consensus algorithm is described as follows:

Initially, each agent $i$ chooses a Slater vector $z_i(0) \in X$ such that $g(z_i(0)) < 0$. At every time $k \geq 0$, each agent $i$ updates its estimates by using the following rule:

$$z_i(k+1) = \max_{j\in\mathcal{N}_i(k)\cup\{i\}} z_j(k). \tag{2}$$

where we use the following relation for vectors: for $a, b \in \mathbb{R}^n$, $a < b$ if and only if there is some $\ell \in \{1, \ldots, n-1\}$ such that $a_\kappa = b_\kappa$ for all $\kappa < \ell$ and $a_\ell < b_\ell$.

The periodical strong connectivity assumption 2.3 ensures that after at most $(N-1)B$ steps, all the agents reach the consensus; i.e., $z_i(k) = \max_{j\in V} z_j(0)$ for all $k \geq (N-1)B$. In the remainder of this paper, we assume that the Slater vector $\bar{z}$ is known to all the agents. •

[1]It is also referred to as double stochasticity.

In [29], in order to solve the convex case of the problem $(P)$ (i.e.; $f_i$ and $g$ are convex functions and $X$ is a convex set), we propose two distributed primal-dual subgradient algorithms where primal (resp. dual) estimates move along subgradients (resp. supgradients) and are projected onto convex sets. The absence of convexity impedes the use of the algorithms in [29] since, on the one hand, (primal) gradient-based algorithms are easily trapped in local minima.; on the other hand, projection maps may not be well-defined when (primal) state constraint sets are non-convex. In this paper, we will employ Lagrangian dualization to circumvent the challenges caused by non-convexity.

Towards this end, we construct a directed cyclic graph $\mathcal{G}_{\rm cyc} := (V, E_{\rm cyc})$ where $|E_{\rm cyc}| = N$. We assume that each agent has a unique in-neighbor (and out-neighbor). The out-neighbor (resp. in-neighbor) of agent $i$ is denoted by $i_D$ (resp. $i_U$). With the graph $\mathcal{G}_{\rm cyc}$, we will study the following approximate problem of problem $(P)$:

$$\begin{aligned} &\min_{(x_i)\in\mathbb{R}^{nN}} \sum_{i\in V} f_i(x_i), \\ &\text{s.t.} \quad g(x_i) \leq 0, \quad , -x_i + x_{i_D} - \Delta \leq 0 \\ &x_i - x_{i_D} - \Delta \leq 0, \quad x_i \in X, \quad \forall i \in V, \end{aligned} \tag{3}$$

where $\Delta := \delta\mathbf{1}$, with $\delta$ a small positive scalar, and $\mathbf{1}$ is the column vector of $n$ ones. The problem (3) reduces to the problem $(P)$ when $\delta = 0$, and will be referred to as problem $(P_\Delta)$. Its optimal value and the set of optimal solutions will be denoted by $p_\Delta^*$ and $X_\Delta^*$, respectively. Similarly to the problem $(P)$, $p_\Delta^*$ is finite and $X_\Delta^* \neq \emptyset$.

**Remark 2.2:** The cyclic graph $\mathcal{G}_{\rm cyc}$ can be replaced by any strongly connected graph. Each agent $i$ is endowed with two inequality constraints: $x_i - x_j - \Delta \leq 0$ and $-x_i + x_j - \Delta \leq 0$, for each out-neighbor $j$. For notational simplicity, we will use the cyclic graph $\mathcal{G}_{\rm cyc}$, which has a minimum number of constraints, as the initial graph. •

### A. *Dual problems*

Before introducing dual problems, let us denote by $\Xi_i := \mathbb{R}^m_{\geq 0} \times \mathbb{R}^{nN}_{\geq 0} \times \mathbb{R}^{nN}_{\geq 0}$, $\Xi := \mathbb{R}^{mN}_{\geq 0} \times \mathbb{R}^{nN}_{\geq 0} \times \mathbb{R}^{nN}_{\geq 0}$, $\xi_i := (\mu_i, \lambda, w) \in \Xi_i$, $\xi := (\mu, \lambda, w) \in \Xi$ and $x := (x_i) \in X^N$. The dual problem $(D_\Delta)$ associated with $(P_\Delta)$ is given by

$$\max_{\mu,\lambda,w} Q(\mu, \lambda, w), \quad \text{s.t.} \quad \mu, \lambda, w \geq 0, \tag{4}$$

where $\mu := (\mu_i) \in \mathbb{R}^{mN}$, $\lambda := (\lambda_i) \in \mathbb{R}^{nN}$ and $w := (w_i) \in \mathbb{R}^{nN}$. Here, the dual function $Q : \Xi \to \mathbb{R}$ is given as

$$Q(\xi) \equiv Q(\mu, \lambda, w) := \inf_{x\in X^N} \mathcal{L}(x, \mu, \lambda, w),$$

where $\mathcal{L} : \mathbb{R}^{nN} \times \Xi \to \mathbb{R}$ is the Lagrangian function

$$\begin{aligned} \mathcal{L}(x, \xi) \equiv \mathcal{L}(x, \mu, \lambda, w) := \sum_{i\in V} \big(f_i(x_i) + \langle \mu_i, g(x_i)\rangle \\ + \langle \lambda_i, -x_i + x_{i_D} - \Delta\rangle + \langle w_i, x_i - x_{i_D} - \Delta\rangle\big). \end{aligned}$$

We denote the dual optimal value of the problem $(D_\Delta)$ by $d_\Delta^*$ and the set of dual optimal solutions by $D_\Delta^*$. In what follows we will assume that the duality gap is zero.

**Assumption 2.5 (Strong duality):** For the introduced problems $(P_\Delta)$ and $(D_\Delta)$, it holds that $p^*_\Delta = d^*_\Delta$.

We endow each agent $i$ with the local Lagrangian function $\mathcal{L}_i : \mathbb{R}^n \times \Xi_i \to \mathbb{R}$ and the local dual function $Q_i : \Xi_i \to \mathbb{R}$ defined by

$$\begin{aligned}\mathcal{L}_i(x_i, \xi_i) &:= f_i(x_i) + \langle \mu_i, g(x_i)\rangle + \langle -\lambda_i + \lambda_{i_U}, x_i\rangle \\ &\quad + \langle w_i - w_{i_U}, x_i\rangle - \langle \lambda_i, \Delta\rangle - \langle w_i, \Delta\rangle, \\ Q_i(\xi_i) &:= \inf_{x_i \in X} \mathcal{L}_i(x_i, \xi_i).\end{aligned}$$

In the problem $(P_\Delta)$, the introduction of approximate consensus constraints $-\Delta \le x_i - x_{i_D} \le \Delta$, $i \in V$, renders the $f_i$ and $g$ separable. As a result, the global dual function $Q$ can be decomposed into a simple sum of the local dual functions $Q_i$. More precisely, the following holds:

$$\begin{aligned}Q(\xi) &= \inf_{x \in X^N} \sum_{i \in V} \big(f_i(x_i) + \langle \mu_i, g(x_i)\rangle \\ &+ \langle \lambda_i, -x_i + x_{i_D} - \Delta\rangle + \langle w_i, x_i - x_{i_D} - \Delta\rangle\big) \\ &= \inf_{x \in X^N} \sum_{i \in V} \big(f_i(x_i) + \langle \mu_i, g(x_i)\rangle \\ &+ \langle -\lambda_i + \lambda_{i_U}, x_i\rangle + \langle w_i - w_{i_U}, x_i\rangle - \langle \lambda_i, \Delta\rangle - \langle w_i, \Delta\rangle\big) \\ &= \sum_{i \in V} \inf_{x_i \in X} \big(f_i(x_i) + \langle \mu_i, g(x_i)\rangle \\ &+ \langle -\lambda_i + \lambda_{i_U}, x_i\rangle + \langle w_i - w_{i_U}, x_i\rangle - \langle \lambda_i, \Delta\rangle - \langle w_i, \Delta\rangle\big) \\ &= \sum_{i \in V} Q_i(\xi_i). \qquad (5)\end{aligned}$$

It is worth mentioning that $\sum_{i \in V} Q_i(\xi_i)$ is not separable since $Q_i$ depends upon neighbor's multipliers $\lambda_{i_U}$ and $w_{i_U}$.

### B. Dual solution sets

The Slater's condition ensures the boundedness of dual solution sets for convex optimization; e.g., [9], [17]. We will shortly see that the Slater's condition plays the same role in non-convex optimization. To achieve this, we define the function $\hat{Q}_i : \mathbb{R}^m_{\ge 0} \times \mathbb{R}^n_{\ge 0} \times \mathbb{R}^n_{\ge 0} \to \mathbb{R}$ as follows:

$$\begin{aligned}\hat{Q}_i(\mu_i, \lambda_i, w_i) &= \inf_{x_i \in X, x_{i_D} \in X} \big(f_i(x_i) + \langle \mu_i, g(x_i)\rangle \\ &+ \langle \lambda_i, -x_i + x_{i_D} - \Delta\rangle + \langle w_i, x_i - x_{i_D} - \Delta\rangle\big).\end{aligned}$$

Let $\bar{z}$ be a Slater vector for problem $(P)$. Then $\bar{x} = (\bar{x}_i) \in X^N$ with $\bar{x}_i = \bar{z}$ is a Slater vector of the problem $(P_\Delta)$. Similarly to (3) and (4) in [29], which make use of Lemma 3.2 in the same paper, we have that for any $\mu_i, \lambda_i, w_i \ge 0$, it holds that

$$\max_{\xi \in D^*_\Delta} \|\xi\| \le N \max_{i \in V} \frac{f_i(\bar{z}) - \hat{Q}_i(\mu_i, \lambda_i, w_i)}{\beta(\bar{z})}, \qquad (6)$$

where $\beta(\bar{z}) := \min\{\min_{\ell \in \{1,\dots,m\}} -g_\ell(\bar{z}), \delta\}$. Let $\mu_i$, $\lambda_i$ and $w_i$ be zero in (6), and it leads to the following upper bound on $D^*_\Delta$:

$$\max_{\xi \in D^*_\Delta} \|\xi\| \le N \max_{i \in V} \frac{f_i(\bar{z}) - \hat{Q}_i(0,0,0)}{\beta(\bar{z})}, \qquad (7)$$

where $\hat{Q}_i(0,0,0) = \inf_{x_i \in X} f_i(x_i)$ and it can be computed locally. Since $f_i$ and $g$ are continuous and $X$ is compact, it is known that $Q_i$ is continuous; e.g., see Theorem 1.4.16 in [1]. Similarly, $Q$ is continuous. Since $D^*_\Delta$ is also bounded, then we have that $D^*_\Delta \ne \emptyset$.

**Remark 2.3:** The requirement of exact agreement on $z$ in the problem $P$ is slightly relaxed in the problem $P_\Delta$ by introducing a small positive scalar $\delta$. In this way, on the one hand, the global dual function $Q$ is a sum of the local dual functions $Q_i$, as in (5); on the other hand, $D^*_\Delta$ is non-empty and uniformly bounded. These two properties play important roles in the devise of our sequent algorithm. •

### C. Other notation

Denote by *the approximate dual optimal solution set* $D^\epsilon_\Delta := \{\xi \in \Xi \mid Q(\xi) \ge d^*_\Delta - N\epsilon\}$. Similar to (7), we have the following upper bound on $D^\epsilon_\Delta$:

$$\max_{\xi \in D^\epsilon_\Delta} \|\xi\| \le N \max_{i \in V} \frac{f_i(\bar{z}) - \hat{Q}_i(0,0,0) + \epsilon}{\beta(\bar{z})}. \qquad (8)$$

In the algorithm we will present in the following section, agents will compute $\gamma_i(\bar{z}) := \frac{f_i(\bar{z}) - \hat{Q}_i(0,0,0)+\epsilon}{\beta(\bar{z})}$.

Define the set-valued map $\Omega_i : \Xi_i \to 2^X$ in the following way $\Omega_i(\xi_i) := \mathrm{argmin}_{x_i \in X} \mathcal{L}_i(x_i, \xi_i)$; i.e., given $\xi_i$, the set $\Omega_i(\xi_i)$ is the collection of solutions to the following local optimization problem:

$$\min_{x_i \in X} \mathcal{L}_i(x_i, \xi_i). \qquad (9)$$

Here, $\Omega_i$ is referred to as the *marginal map* of agent $i$. Since $X$ is compact and $f_i$, $g$ are continuous, then $\Omega_i(\xi_i) \ne \emptyset$ in (9) for any $\xi_i \in \Xi_i$. In the algorithm we will develop in next section, each agent is required to solve the local optimization problem (9) at each iterate. We assume that this problem (9) can be easily solved. This is the case for problems of $n = 1$, or $f_i$ and $g$ being smooth (the extremum candidates are the critical points of the objective function and isolated corners of the boundaries of the constraint regions) or having some specific structure which allows the use of global optimization methods such as branch and bound algorithms. For some $\epsilon > 0$, we define the set-valued map $\Omega^\epsilon_i : \Xi_i \to 2^X$ as follows:

$$\Omega^\epsilon_i(\xi_i) := \{x_i \in X \mid \mathcal{L}_i(x_i, \xi_i) \le Q_i(\xi_i) + \epsilon\},$$

which is referred to as the *approximate marginal map* of agent $i \in V$.

In the space $\mathbb{R}^n$, we define the distance between a point $z \in \mathbb{R}^n$ to a set $A \subset \mathbb{R}^n$ as $\mathrm{dist}(z, A) := \inf_{y \in A} \|z - y\|$, and the Hausdorff distance between two sets $A, B \subset \mathbb{R}^n$ as $\mathrm{dist}(A, B) := \max\{\sup_{z \in A} \mathrm{dist}(z, B), \sup_{y \in B} \mathrm{dist}(A, y)\}$. We denote by $B_{\mathcal{U}}(A, r) := \{u \in \mathcal{U} \mid \mathrm{dist}(u, A) \le r\}$ and $B_{2^{\mathcal{U}}}(A, r) := \{U \in 2^{\mathcal{U}} \mid \mathrm{dist}(U, A) \le r\}$ where $\mathcal{U} \subset \mathbb{R}^n$.

## III. Distributed approximate dual subgradient algorithm

In this section, we devise a distributed approximate dual subgradient algorithm which aims to find a pair of approximate primal-dual solutions to the problem $(P_\Delta)$. Its convergence properties are also summarized.

For each agent $i$, let $x_i(k) \in \mathbb{R}^n$ be the estimate of the primal solution $x_i$ to the problem $(P_\Delta)$ at time $k \geq 0$, $\mu_i(k) \in \mathbb{R}^m_{\geq 0}$ be the estimate of the multiplier on the inequality constraint $g(x_i) \leq 0$, $\lambda^i(k) \in \mathbb{R}^{nN}_{\geq 0}$ (resp. $w^i(k) \in \mathbb{R}^{nN}_{\geq 0}$)[2] be the estimate of the multiplier associated with the collection of the local inequality constraints $-x_j + x_{j_D} - \Delta \leq 0$ (resp. $x_j - x_{j_D} - \Delta \leq 0$), for all $j \in V$. We let $\xi_i(k) := (\mu_i(k)^T, \lambda^i(k)^T, w^i(k)^T)^T$, for $i \in V$, and $v_i(k) := (\mu_i(k)^T, v^i_\lambda(k)^T, v^i_w(k)^T)^T$ where $v^i_\lambda(k) := \sum_{j\in V} a^i_j(k)\lambda^j(k)$ and $v^i_w(k) := \sum_{j\in V} a^i_j(k) w^j(k)$.

The *Distributed Approximate Dual Subgradient* (DADS, for short) Algorithm is described as follows:

Initially, each agent $i$ chooses a common Slater vector $\bar{z}$, computes $\gamma_i(\bar{z})$ and obtains $\gamma := N \max_{i\in V} \gamma_i(\bar{z})$ through a max-consensus algorithm. After that, each agent $i$ chooses initial states $x_i(0) \in X$ and $\xi_i(0) \in \Xi_i$.

Agent $i$ updates $x_i(k)$ and $\xi_i(k)$ as follows:

Step 1. For each $k \geq 1$, given $v_i(k)$, solve the local optimization problem (9), obtain a solution in $\Omega_i(v_i(k))$ and the dual optimal value $Q_i(v_i(k))$. Produce the primal estimate $x_i(k)$ in the following way: if $x_i(k-1) \in \Omega^\epsilon_i(v_i(k))$, then $x_i(k) = x_i(k-1)$; otherwise, choose $x_i(k) \in \Omega_i(v_i(k))$.

Step 2. For each $k \geq 0$, generate the dual estimate $\xi_i(k+1)$ according to the following rule:

$$\xi_i(k+1) = P_{M_i}[v_i(k) + \alpha(k)\mathcal{D}_i(k)], \qquad (10)$$

where the scalar $\alpha(k)$ is a step-size. The supgradient vector of agent $i$ is defined as $\mathcal{D}_i(k) := (\mathcal{D}^i_\mu(k)^T, \mathcal{D}^i_\lambda(k)^T, \mathcal{D}^i_w(k)^T)^T$, where $\mathcal{D}^i_\mu(k) := g(x_i(k)) \in \mathbb{R}^m$, $\mathcal{D}^i_\lambda(k)$ has components $\mathcal{D}^i_\lambda(k)_i := -\Delta - x_i(k) \in \mathbb{R}^n$, $\mathcal{D}^i_\lambda(k)_{i_U} := x_i(k) \in \mathbb{R}^n$, and $\mathcal{D}^i_\lambda(k)_j = 0 \in \mathbb{R}^n$ for $j \in V \setminus \{i, i_U\}$, while the components of $\mathcal{D}^i_w(k)$ are given by: $\mathcal{D}^i_w(k)_i := -\Delta + x_i(k) \in \mathbb{R}^n$, $\mathcal{D}^i_w(k)_{i_U} := -x_i(k) \in \mathbb{R}^n$, and $\mathcal{D}^i_w(k)_j = 0 \in \mathbb{R}^n$, for $j \in V \setminus \{i, i_U\}$. The set $M_i$ in the projection map, $P_{M_i}$, above is defined as $M_i := \{\xi_i \in \Xi_i \mid \|\xi_i\| \leq \gamma + \theta\}$ for some $\theta > 0$.

**Remark 3.1:** In the initialization of the DADS algorithm, the quantity $\gamma$ is an upper bound on $D^\epsilon_\Delta$. Note that in Step 1, the check $x_i(k-1) \in \Omega^\epsilon_i(v_i(k))$ reduces to verifying that $\mathcal{L}_i(x_i(k-1), v_i(k)) \leq Q_i(v_i(k)) + \epsilon$. Then, only if $\mathcal{L}_i(x_i(k-1), v_i(k)) > Q_i(v_i(k)) + \epsilon$, it is necessary to find *one solution* in $\Omega_i(v_i(k))$. That is, it is unnecessary to compute all the set $\Omega_i(v_i(k))$. In Step 2, since $M_i$ is closed and convex, the projection map $P_{M_i}$ is well-defined. •

The primal and dual estimates in the DADS algorithm will be shown to asymptotically converge to a pair of approximate primal-dual solutions to the problem $(P_\Delta)$. We formally state this in the following.

**Theorem 3.1:** Consider the problem $(P)$ and the corresponding approximate problem $(P_\Delta)$ with some $\delta > 0$. We let the non-degeneracy assumption 2.1, the balanced communication assumption 2.2 and the periodic strong connectivity assumption 2.3 hold. In addition, suppose the Slater's condition 2.4 holds for the problem $(P)$ and the strong duality assumption 2.5 holds for the problem $(P_\Delta)$. Consider the dual sequences of $\{\mu_i(k)\}$, $\{\lambda^i(k)\}$, $\{w^i(k)\}$ and the primal sequence of $\{x_i(k)\}$ of the distributed approximate dual subgradient algorithm with the step-sizes $\{\alpha(k)\}$ satisfying $\lim_{k\to+\infty} \alpha(k) = 0$, $\sum_{k=0}^{+\infty} \alpha(k) = +\infty$, and $\sum_{k=0}^{+\infty} \alpha(k)^2 < +\infty$. Then, there exists a feasible dual pair $\tilde{\xi} := (\tilde{\mu}, \tilde{\lambda}, \tilde{w})$ such that $\lim_{k\to+\infty} \|\mu_i(k) - \tilde{\mu}_i\| = 0$, $\lim_{k\to+\infty} \|\lambda^i(k) - \tilde{\lambda}\| = 0$, and $\lim_{k\to+\infty} \|w^i(k) - \tilde{w}\| = 0$, for all $i \in V$. Moreover, there is a feasible primal vector $\tilde{x} := (\tilde{x}_i) \in X^N$ such that $\lim_{k\to+\infty} \|x_i(k) - \tilde{x}_i\| = 0$, for all $i \in V$. In addition, $(\tilde{x}, \tilde{\xi})$ is a pair of approximate primal-dual solutions in the sense that $d^*_\Delta - N\epsilon \leq Q(\tilde{\xi}) \leq d^*_\Delta = p^*_\Delta \leq \sum_{i\in V} f_i(\tilde{x}_i) \leq p^*_\Delta + N\epsilon$.

The analysis of Theorem 3.1 will be provided in next section. Before doing that, we would like to discuss several possible extensions of Theorem 3.1.

Firstly, the step-size scheme in the DADS algorithm can be slightly generalized to the following: $\lim_{k\to+\infty} \alpha_i(k) = 0$, $\sum_{k=0}^{+\infty} \alpha_i(k) = +\infty$, $\sum_{k=0}^{+\infty} \alpha_i(k)^2 < +\infty$, $\min_{i\in V} \alpha_i(k) \geq C_\alpha \max_{i\in V} \alpha_i(k)$, where $\alpha_i(k)$ is the step-size of agent $i$ at time $k$ and $C_\alpha \in (0, 1]$.

Secondly, the periodic strong connectivity assumption 2.3 can be weakened into the eventual strong connectivity assumption, e.g. Assumption 6.1 in [29], if $\mathcal{G}(k)$ is undirected.

Thirdly, each agent can use a different $\epsilon_i$ in Step 1 of the DADS algorithm, which would lead to replacing $N\epsilon$ in the approximate solution by $\sum_{i\in V} \epsilon_i$.

Lastly, each agent $i$ could have different constraint functions $g_i$ and constraint sets $X_i$ if a Slater vector is known to all the agents. For example, consider the case that $g$ is convex, $X_i$ is convex and potentially different, and there is a Slater vector $\bar{z} \in \cap_{i\in V} X_i$. Then the solution $\tilde{z}$ to the following problem is such that $g(\tilde{z}) \leq g(\bar{z}) < 0$:

$$\min_{z\in\mathbb{R}^n} Ng(z), \quad \text{s.t.} \quad z \in X_i, \ \ \forall i \in V \qquad (11)$$

Through implementing the distributed primal subgradient algorithm in [29], agents can solve the problem (11) in a distributed fashion and agree upon the minimizer $\tilde{z}$ which coincides with a Slater vector. In such a way, Theorem 3.1 still holds and the corresponding proof is a slight variation of those in next section.

## IV. Convergence analysis

Recall that $g$ is continuous and $X$ is compact. Then there are $G, H > 0$ such that $\|g(z)\| \leq G$ and $\|z\| \leq H$ for all $z \in X$. We start our analysis of the DADS algorithm from the computation of supgradients of $Q_i$.

**Lemma 4.1 (Approximate supgradient):** If $\bar{x}_i \in \Omega^\epsilon_i(\bar{\xi}_i)$, then $\big(g(\bar{x}_i)^T, (-\Delta - \bar{x}_i)^T, \bar{x}_i^T, (\bar{x}_i - \Delta)^T, -\bar{x}_i^T\big)^T$ is an approximate supgradient of $Q_i$ at $\bar{\xi}_i$; i.e., the following holds for any $\xi_i \in \Xi_i$:

$$\begin{aligned} &Q_i(\xi_i) - Q_i(\bar{\xi}_i) \leq \langle g(\bar{x}_i), \mu_i - \bar{\mu}_i\rangle + \langle -\Delta - \bar{x}_i, \lambda_i - \bar{\lambda}_i\rangle \\ &+ \langle \bar{x}_i, \lambda_{i_U} - \bar{\lambda}_{i_U}\rangle + \langle \bar{x}_i - \Delta, w_i - \bar{w}_i\rangle \\ &+ \langle -\bar{x}_i, w_{i_U} - \bar{w}_{i_U}\rangle + \epsilon. \end{aligned} \qquad (12)$$

[2]We will use the superscript $i$ to indicate that $\lambda^i(k)$ and $w^i(k)$ are estimates of some global variables.

*Proof:* The proof is analogous to the computation of dual subgradients, e.g., in [2], [3], and omitted here due to the space limitation. ■

Since $\Omega_i(v_i(k)) \subseteq \Omega_i^\epsilon(v_i(k))$, it is clear that $x_i(k) \in \Omega_i^\epsilon(v_i(k))$ for all $k \geq 0$. A direct result of Lemma 4.1 is that the vector $(g(x_i(k))^T, (-\Delta - x_i(k))^T, x_i(k)^T, (x_i(k) - \Delta)^T, -x_i(k)^T)$ is an approximate supgradient of $Q_i$ at $v_i(k)$; i.e., the following approximate supgradient inequality holds for any $\xi_i \in \Xi_i$:

$$\begin{aligned}
&Q_i(\xi_i) - Q_i(v_i(k)) \leq \langle g(x_i(k)), \mu_i - \mu_i(k)\rangle \\
&+ \langle -\Delta - x_i(k), \lambda_i - v_\lambda^i(k)_i\rangle \\
&+ \langle x_i(k), \lambda_{i_U} - v_\lambda^i(k)_{i_U}\rangle + \langle x_i(k) - \Delta, w_i - v_w^i(k)_i\rangle \\
&+ \langle -x_i(k), w_{i_U} - v_w^i(k)_{i_U}\rangle + \epsilon. \qquad (13)
\end{aligned}$$

Now we can see that the update rule of dual estimates in the DADS algorithm is a combination of an approximate dual subgradient scheme and average consensus algorithms. The following establishes that $Q_i$ is Lipschitz continuous with some Lipschitz constant $L$.

**Lemma 4.2 (Lipschitz continuity of $Q_i$):** There is a constant $L > 0$ such that for any $\xi_i, \bar{\xi}_i \in \Xi_i$, it holds that

$$\|Q_i(\xi_i) - Q_i(\bar{\xi}_i)\| \leq L\|\xi_i - \bar{\xi}_i\|.$$

*Proof:* Similarly to Lemma 4.1, one can show that if $\bar{x}_i \in \Omega_i(\bar{\xi}_i)$, then $(g(\bar{x}_i)^T, (-\Delta - \bar{x}_i)^T, \bar{x}_i^T, (\bar{x}_i - \Delta)^T, -\bar{x}_i^T)^T$ is a supgradient of $Q_i$ at $\bar{\xi}_i$; i.e., the following holds for any $\xi_i \in \Xi_i$:

$$\begin{aligned}
&Q_i(\xi_i) - Q_i(\bar{\xi}_i) \leq \langle g(\bar{x}_i), \mu_i - \bar{\mu}_i\rangle + \langle -\Delta - \bar{x}_i, \lambda_i - \bar{\lambda}_i\rangle \\
&+ \langle \bar{x}_i, \lambda_{i_U} - \bar{\lambda}_{i_U}\rangle + \langle \bar{x}_i - \Delta, w_i - \bar{w}_i\rangle \\
&+ \langle -\bar{x}_i, w_{i_U} - \bar{w}_{i_U}\rangle.
\end{aligned}$$

Since $\|g(\bar{x}_i)\| \leq G$ and $\|\bar{x}_i\| \leq H$, there is $L > 0$ such that $Q_i(\xi_i) - Q_i(\bar{\xi}_i) \leq L\|\xi_i - \bar{\xi}_i\|$. Similarly, $Q_i(\bar{\xi}_i) - Q_i(\xi_i) \leq L\|\xi_i - \bar{\xi}_i\|$. The combination of these two relations renders the desired result. ■

In the DADS algorithm, the error induced by the projection map $P_{M_i}$ is given by:

$$e_i(k) := P_{M_i}[v_i(k) + \alpha(k)\mathcal{D}_i(k)] - v_i(k).$$

We next provide a basic iterate relation of dual estimates in the DADS algorithm.

**Lemma 4.3 (Basic iterate relation):** Under the assumptions in Theorem 3.1, for any $((\mu_i), \lambda, w) \in \Xi$ with $(\mu_i, \lambda, w) \in M_i$ for all $i \in V$, the following estimate holds for all $k \geq 0$:

$$\begin{aligned}
&\sum_{i\in V}\|e_i(k) - \alpha(k)\mathcal{D}_i(k)\|^2 \leq \alpha(k)^2\sum_{i\in V}\|\mathcal{D}_i(k)\|^2 \\
&+ \sum_{i\in V}(\|\xi_i(k) - \xi_i\|^2 - \|\xi_i(k+1) - \xi_i\|^2) \\
&+ 2\alpha(k)\sum_{i\in V}\{\langle g(x_i(k)), \mu_i(k) - \mu_i\rangle \\
&+ \langle -\Delta - x_i(k), v_\lambda^i(k)_i - \lambda_i\rangle \\
&+ \langle x_i(k), v_\lambda^i(k)_{i_U} - \lambda_{i_U}\rangle + \langle x_i(k) - \Delta, v_w^i(k)_i - w_i\rangle \\
&+ \langle -x_i(k), v_w^i(k)_{i_U} - w_{i_U}\rangle\}. \qquad (14)
\end{aligned}$$

*Proof:* Recall that $M_i$ is closed and convex. The proof is an application of Lemma 7.1 in the Appendix. ■

The lemma below shows that dual estimates asymptotically converge to some approximate dual optimal solution.

**Lemma 4.4 (Dual estimate convergence):** Under the assumptions in Theorem 3.1, there exist a feasible dual pair $\tilde{\xi} := ((\tilde{\mu}_i), \tilde{\lambda}, \tilde{w})$ such that $\lim_{k\to+\infty}\|\mu_i(k) - \tilde{\mu}_i\| = 0$, $\lim_{k\to+\infty}\|\lambda^i(k) - \tilde{\lambda}\| = 0$, and $\lim_{k\to+\infty}\|w^i(k) - \tilde{w}\| = 0$. Furthermore, the vector $\tilde{\xi}$ is an approximate dual solution to the problem $(D_\Delta)$ in the sense that $d_\Delta^* - N\epsilon \leq Q(\tilde{\xi}) \leq d_\Delta^*$.

*Proof:* By the dual decomposition property (5) and the boundedness of dual optimal solution sets, the dual problem $(D_\Delta)$ is equivalent to the following:

$$\max_{(\xi_i)}\sum_{i\in V}Q_i(\xi_i), \quad \text{s.t.} \quad \xi_i \in M_i. \qquad (15)$$

Note that $Q_i$ is affine and $M_i$ is convex, implying that the problem (15) is a constrained convex programming where the global objective function is a simple sum of local ones and the local state constraints are compact.

Since $X$ and $M_i$ are compact, there is some $J > 0$ which is an upper bound of the norm of the last sum on the right-hand side of (14). In this way, inequality (14) leads to:

$$\begin{aligned}
&\sum_{i\in V}\|\xi_i(K) - \xi_i\|^2 \leq \sum_{i\in V}\|\xi_i(K') - \xi_i\|^2 \\
&+ \alpha(K')^2\sum_{i\in V}\|\mathcal{D}_i(K')\|^2 + 2\alpha(K')J, \qquad (16)
\end{aligned}$$

where $K = K' + 1$. It is not difficult to see that the sequence of $\{\mathcal{D}_i(k)\}$ is uniformly bounded. Since $\lim_{k\to+\infty}\alpha(k) = 0$, then we take the limits on $K$, and $K'$ in (16), and it renders that $\limsup_{K\to+\infty}\sum_{i\in V}\|\xi_i(K) - \xi_i\|^2 \leq \liminf_{K'\to+\infty}\sum_{i\in V}\|\xi_i(K') - \xi_i\|^2$. Therefore, we have $\lim_{k\to+\infty}\sum_{i\in V}\|\xi_i(k) - \xi_i\|^2$ exists.

By using this property and taking the limit on both sides of (14), we then have $\lim_{k\to+\infty}\|e_i(k)\| = 0$. By using Proposition 7.1 in the Appendix, we conclude that the consensus on $\lambda$ and $w$ is asymptotically achieved; i.e., $\lim_{k\to+\infty}\|\lambda^i(k) - \lambda^j(k)\| = 0$ and $\lim_{k\to+\infty}\|w^i(k) - w^j(k)\| = 0$ for any $i, j \in V$. Combining these with the convergence of $\{\sum_{i\in V}\|\xi_i(k) - \xi_i\|^2\}$ and the closedness of $M_i$, we can

deduce that there exist a feasible dual pair $\tilde{\xi} := ((\tilde{\mu}_i), \tilde{\lambda}, \tilde{w})$ such that $\lim_{k\to+\infty} \|\mu_i(k) - \tilde{\mu}_i\| = 0$, $\lim_{k\to+\infty} \|\lambda^i(k) - \tilde{\lambda}\| = 0$, and $\lim_{k\to+\infty} \|w^i(k) - \tilde{w}\| = 0$, for all $i \in V$. Furthermore, we have $Q(\tilde{\xi}) \le d^*_\Delta$.

Substitute the approximate supgradient inequality (13) into (14), rearrange terms, and we have

$$2\alpha(k)\sum_{i\in V}(Q_i(\xi_i) - Q_i(v_i(k)) - \epsilon) \le \sum_{i\in V}\alpha(k)^2\|\mathcal{D}_i(k)\|^2 + \sum_{i\in V}(\|\xi_i(k) - \xi_i\|^2 - \|\xi_i(k+1) - \xi_i\|^2). \tag{17}$$

Let $\hat{\lambda}(k) := \frac{1}{N}\sum_{i\in V}\lambda^i(k)$ and $\hat{w}(k) := \frac{1}{N}\sum_{i\in V} w^i(k)$. By Lipschitz continuity of $Q_i$, it follows from (17) that

$$\begin{aligned}&\sum_{i\in V} 2\alpha(k)(Q_i(\xi_i) - Q_i(\mu_i(k), \hat{\lambda}(k), \hat{w}(k)) - \epsilon)\\ &\le \sum_{i\in V}\alpha(k)^2\|\mathcal{D}_i(k)\|^2\\ &+ \sum_{i\in V}(\|\xi_i(k) - \xi_i\|^2 - \|\xi_i(k+1) - \xi_i\|^2)\\ &+ \sum_{i\in V} 2\alpha(k)L(\|v^i_\lambda(k) - \hat{\lambda}(k)\| + \|v^i_w(k) - \hat{w}(k)\|).\end{aligned} \tag{18}$$

Now we follow a contradiction argument, and state $\tilde{\xi}$ is not approximate dual optimal. That is, assume that $\sum_{i\in V} Q_i(\tilde{\mu}_i, \tilde{\lambda}, \tilde{w}) < d^*_\Delta - N\epsilon$. Then $\rho := -\sum_{i\in V} Q_i(\tilde{\mu}_i, \tilde{\lambda}, \tilde{w}) + d^*_\Delta - N\epsilon > 0$. Let $\xi_i$ in (18) be some dual optimal solution. Since $\lim_{k\to+\infty}\|v^i_\lambda(k) - \hat{\lambda}(k)\| = 0$ and $\lim_{k\to+\infty}\|v^i_w(k) - \hat{w}(k)\| = 0$, there is $K' \ge 0$ such that for all $k \ge K'$, there holds

$$\begin{aligned}\frac{1}{2}\rho\alpha(k) &\le \sum_{i\in V}\alpha(k)^2\|\mathcal{D}_i(k)\|^2\\ &+ \sum_{i\in V}(\|\xi_i(k) - \xi_i\|^2 - \|\xi_i(k+1) - \xi_i\|^2)\end{aligned} \tag{19}$$

Sum (19) over $[K', K]$ and rearrange it. It gives that

$$\begin{aligned}\sum_{i\in V}\|\xi_i(K+1) - \xi_i\|^2 &\le \sum_{k=K'}^{K}\sum_{i\in V}\alpha(k)^2\|\mathcal{D}_i(k)\|^2\\ &- \frac{1}{2}\rho\sum_{k=K'}^{K}\alpha(k) + \sum_{i\in V}\|\xi_i(K') - \xi_i\|^2\end{aligned}$$

Since $\{\xi_i(k)\}$ converges, it is uniformly bounded. Recall that $\{\alpha(k)\}$ is not summable but square summable. When $K$ is sufficiently large, the above inequality leads to a contradiction. Hence, it must be that $d^*_\Delta - N\epsilon \le Q(\tilde{\xi})$. ∎

The remainder of this section is dedicated to characterizing the convergence properties of primal estimates. Toward this end, we present the closedness and upper semicontinuity properties of $\Omega^\epsilon_i$.

**Lemma 4.5 (Properties of $\Omega^\epsilon_i$):** The approximate set-valued marginal map $\Omega^\epsilon_i$ is closed. In addition, it is upper semicontinuous at $\xi_i \in \Xi_i$; i.e., for any $\epsilon' > 0$, there is $\delta > 0$ such that for any $\tilde{\xi}_i \in B_{\Xi_i}(\xi_i, \delta)$, it holds that $\Omega^\epsilon_i(\tilde{\xi}_i) \subset B_{2^X}(\Omega^\epsilon_i(\xi_i), \epsilon')$.

*Proof:* Consider sequences $\{x_i(k)\}$ and $\{\xi_i(k)\}$ satisfying $\lim_{k\to+\infty}\xi_i(k) = \bar{\xi}_i$, $x_i(k) \in \Omega^\epsilon_i(\xi_i(k))$ and $\lim_{k\to+\infty} x_i(k) = \bar{x}_i$. Since $\mathcal{L}_i$ is continuous, then we have

$$\begin{aligned}\mathcal{L}_i(\bar{x}_i, \bar{\xi}_i) &= \lim_{k\to+\infty}\mathcal{L}_i(x_i(k), \xi_i(k))\\ &\le \lim_{k\to+\infty}(Q_i(\xi_i(k)) + \epsilon) = Q_i(\bar{\xi}_i) + \epsilon,\end{aligned}$$

where in the inequality we use the property of $x_i(k) \in \Omega^\epsilon_i(\xi_i(k))$, and in the last equality we use the continuity of $Q_i$. Then $\bar{x}_i \in \Omega^\epsilon_i(\bar{\xi}_i)$ and the closedness of $\Omega^\epsilon_i$ follows.

Note that $\Omega^\epsilon_i(\xi_i) = \Omega^\epsilon_i(\xi_i) \cap X$. Recall that $\Omega^\epsilon_i$ is closed and $X$ is compact. Then it is a result of Theorem 7.1 in the Appendix that $\Omega^\epsilon_i(\xi_i)$ is upper semicontinuous at $\xi_i \in \Xi_i$; i.e, for any neighborhood $\mathcal{U}$ in $2^X$ of $\Omega^\epsilon_i(\xi_i)$, there is $\delta > 0$ such that $\forall\tilde{\xi}_i \in B_{\Xi_i}(\xi_i, \delta)$, it holds that $\Omega^\epsilon_i(\tilde{\xi}_i) \subset \mathcal{U}$. Let $\mathcal{U} = B_{2^X}(\Omega^\epsilon_i(\xi_i), \epsilon')$, and we obtain the property of upper semicontinuity at $\xi_i$. ∎

Upper semicontinuity of $\Omega^\epsilon_i$ ensures that each accumulation point of $\{x_i(k)\}$ is a point in the set $\Omega^\epsilon_i(\tilde{\xi}_i)$; i.e., the convergence of $\{x_i(k)\}$ to the set $\Omega^\epsilon_i(\tilde{\xi}_i)$ can be guaranteed. In what follows, we further characterize the convergence of $\{x_i(k)\}$ to a point in $\Omega^\epsilon_i(\tilde{\xi}_i)$ within a finite time.

**Lemma 4.6 (Primal estimate convergence):** For each $i \in V$, there are a finite time $T_i \ge 0$ and $\tilde{x}_i \in \Omega^\epsilon_i(\tilde{\xi}_i)$ such that $x_i(k) = \tilde{x}_i$ for all $k \ge T_i + 1$.

*Proof:* Choose $\bar{\epsilon} > 0$ and $\hat{\epsilon} > 0$ such that $2(G + 4H + 2\sqrt{m}\delta)\bar{\epsilon} + 2\hat{\epsilon} \le \epsilon$. Since $Q_i$ is continuous and $\lim_{k\to+\infty}\|v_i(k) - \tilde{\xi}_i\| = 0$, then there is $K_i \ge 0$ such that for all $k \ge K_i$, it holds that

$$\|\tilde{\xi}_i - v_i(k)\| \le \bar{\epsilon}, \quad \|Q_i(\tilde{\xi}_i) - Q_i(v_i(k))\| \le \hat{\epsilon}. \tag{20}$$

The time instant $T_i \ge 0$ is defined as follows: if there is some finite time $k \ge K_i + 1$ such that $\mathcal{L}_i(x_i(k), v_i(k+1)) > Q_i(v_i(k+1)) + \epsilon$, then $T_i$ is the smallest one among such $k$; otherwise, $T_i = K_i + 1$. In what follows we prove that $T_i$ is the time in the statement of the lemma.

Consider the first case of $T_i$. In this case, $\mathcal{L}_i(x_i(T_i), v_i(T_i + 1)) > Q_i(v_i(T_i + 1)) + \epsilon$; i.e., $x_i(T_i) \notin \Omega^\epsilon_i(v_i(T_i + 1))$. Then $x_i(T_i + 1) \in \Omega_i(v_i(T_i + 1))$; i.e., $\mathcal{L}_i(x_i(T_i + 1), v_i(T_i + 1)) = Q_i(v_i(T_i + 1))$. By using this property, we have that for any $k \ge T_i + 1$, it holds that

$$\begin{aligned}&\|\mathcal{L}_i(x_i(T_i + 1), v_i(k)) - Q_i(\tilde{\xi}_i)\|\\ &\le \|\mathcal{L}_i(x_i(T_i + 1), v_i(k)) - Q_i(v_i(T_i + 1))\|\\ &+ \|Q_i(v_i(T_i + 1)) - Q_i(\tilde{\xi}_i)\|\\ &= \|\mathcal{L}_i(x_i(T_i + 1), v_i(k)) - \mathcal{L}_i(x_i(T_i + 1), v_i(T_i + 1))\|\\ &+ \|Q_i(v_i(T_i + 1)) - Q_i(\tilde{\xi}_i)\|.\end{aligned} \tag{21}$$

Notice that the term $\|\mathcal{L}_i(x_i(T_i+1), v_i(k)) - \mathcal{L}_i(x_i(T_i+1), v_i(T_i+1))\|$ can be upper bounded in the following way:

$$\begin{aligned}
&\|\mathcal{L}_i(x_i(T_i+1), v_i(k)) - \mathcal{L}_i(x_i(T_i+1), v_i(T_i+1))\| \\
&\leq \|\langle \mu_i(k) - \mu_i(T_i+1), g(x_i(T_i+1))\rangle \\
&+ \langle -v_\lambda^i(k)_i + v_\lambda^i(k)_{i_U} + v_\lambda^i(T_i+1)_i - v_\lambda^i(T_i+1)_{i_U}, \\
&x_i(T_i+1)\rangle + \langle v_w^i(k)_i - v_w^i(k)_{i_U} \\
&- v_w^i(T_i+1)_i + v_w^i(T_i+1)_{i_U}, x_i(T_i+1)\rangle \\
&- \langle v_\lambda^i(k)_i - v_\lambda^i(T_i+1)_i, \Delta\rangle - \langle v_w^i(k)_i - v_w^i(T_i+1)_i, \Delta\rangle\| \\
&\leq 2(G + 4H + 2\sqrt{m}\delta)\bar{\epsilon}. \qquad (22)
\end{aligned}$$

Substituting (20) and (22) into (21) gives that

$$\begin{aligned}
&\|\mathcal{L}_i(x_i(T_i+1), v_i(k)) - Q_i(\tilde{\xi}_i)\| \\
&\leq 2(G + 4H + 2\sqrt{m}\delta)\bar{\epsilon} + \hat{\epsilon}. \qquad (23)
\end{aligned}$$

This implies that for any $k \geq T_i + 1$, it holds that

$$\begin{aligned}
&0 \leq \mathcal{L}_i(x_i(T_i+1), v_i(k)) - Q_i(v_i(k)) \\
&\leq \|\mathcal{L}_i(x_i(T_i+1), v_i(k)) - Q_i(\tilde{\xi}_i)\| \\
&+ \|Q_i(\tilde{\xi}_i) - Q_i(v_i(k))\| \\
&\leq 2(G + 4H + 2\sqrt{m}\delta)\bar{\epsilon} + 2\hat{\epsilon} \leq \epsilon.
\end{aligned}$$

Hence, we conclude that $x_i(T_i+1) \in \Omega_i^\epsilon(v_i(k))$ for all $k \geq T_i + 1$, and thus $x_i(k) = x_i(T_i+1)$ for all $k \geq T_i + 1$.

We now consider the second possibility for $T_i$. In this case, $\mathcal{L}_i(x_i(k), v_i(k+1)) \leq Q_i(v_i(k+1)) + \epsilon$ for all $k \geq T_i = K_i + 1$. Therefore, we have $x_i(T_i+1) \in \Omega_i^\epsilon(v_i(k))$ and then $x_i(k) = x_i(T_i+1)$ for all $k \geq T_i + 1$.

In both cases, the chosen finite $T_i \geq 0$ guarantees that for all $k \geq T_i+1$, $x_i(k) = x_i(T_i+1)$ and $x_i(k) \in \Omega_i^\epsilon(v_i(T_i+1))$. Upper semicontinuity of $\Omega_i^\epsilon$ ensures $x_i(T_i+1) \in \Omega_i^\epsilon(\tilde{\xi}_i)$. ■

Now we are ready to show the main result of this paper, Theorem 3.1. In particular, we will show the property of complementary slackness, primal feasibility of $\tilde{x}$, and characterize its primal suboptimality.

**Proof for Theorem 3.1:**

**Claim 1:** $\langle -\Delta - \tilde{x}_i + \tilde{x}_{i_D}, \tilde{\lambda}_i\rangle = 0$, $\langle -\Delta + \tilde{x}_i - \tilde{x}_{i_D}, \tilde{w}_i\rangle = 0$ and $\langle g(\tilde{x}_i), \tilde{\mu}_i\rangle = 0$.

*Proof:* Rearranging the terms related to $\lambda$ in (14) leads to the following inequality holding for any $((\mu_i), \lambda, w) \in \Xi$ with $(\mu_i, \lambda, w) \in M_i$ for all $i \in V$:

$$\begin{aligned}
&-\sum_{i\in V} 2\alpha(k)(\langle -\Delta - x_i(k), v_\lambda^i(k)_i - \lambda_i\rangle \\
&+ \langle x_{i_D}(k), v_\lambda^{i_D}(k)_i - \lambda_i\rangle) \leq \alpha(k)^2 \sum_{i\in V} \|\mathcal{D}_i(k)\|^2 \\
&+ \sum_{i\in V}(\|\xi_i(k) - \xi_i\|^2 - \|\xi_i(k+1) - \xi_i\|^2) \\
&+ 2\alpha(k)\sum_{i\in V}\{\langle -x_i(k), v_w^i(k)_{i_U} - w_{i_U}\rangle + \langle x_i(k) - \Delta, \\
&v_w^i(k)_i - w_i\rangle + \langle g(x_i(k)), \mu_i(k) - \mu_i\rangle\}. \qquad (24)
\end{aligned}$$

Sum (24) over $[0, K]$, divide by $s(K) := \sum_{k=0}^K \alpha(k)$, and we have

$$\begin{aligned}
&\frac{1}{s(K)}\sum_{k=0}^K \alpha(k)\sum_{i\in V} 2(\langle \Delta + x_i(k), v_\lambda^i(k)_i - \lambda_i\rangle \\
&+ \langle -x_{i_D}(k), v_\lambda^{i_D}(k)_i - \lambda_i\rangle) \leq \frac{1}{s(K)}\sum_{k=0}^K \alpha(k)^2 \sum_{i\in V}\|\mathcal{D}_i(k)\|^2 \\
&+ \frac{1}{s(K)}\{\sum_{i\in V}(\|\xi_i(0) - \xi_i\|^2 - \|\xi_i(K+1) - \xi_i\|^2) \\
&+ \sum_{k=0}^K 2\alpha(k)\sum_{i\in V}(\langle g(x_i(k)), \mu_i(k) - \mu_i\rangle + \langle x_i(k) - \Delta, \\
&v_w^i(k)_i - w_i\rangle + \langle -x_i(k), v_w^i(k)_{i_U} - w_{i_U}\rangle)\}. \qquad (25)
\end{aligned}$$

We now proceed to show $\langle -\Delta - \tilde{x}_i + \tilde{x}_{i_D}, \tilde{\lambda}_i\rangle \geq 0$ for each $i \in V$. Notice that we have shown that $\lim_{k\to+\infty}\|x_i(k) - \tilde{x}_i\| = 0$ for all $i \in V$, and it also holds that $\lim_{k\to+\infty}\|\xi_i(k) - \tilde{\xi}_i\| = 0$ for all $i \in V$. Let $\lambda_i = \frac{1}{2}\tilde{\lambda}_i$, $\lambda_j = \tilde{\lambda}_j$ for $j \neq i$ and $\mu_i = \tilde{\mu}_i$, $w = \tilde{w}$ in (25). Recall that $\{\alpha(k)\}$ is not summable but square summable, and $\{\mathcal{D}_i(k)\}$ is uniformly bounded. Take $K \to +\infty$, and then it follows from Lemma 7.2 in the Appendix that:

$$\langle \Delta + \tilde{x}_i - \tilde{x}_{i_D}, \tilde{\lambda}_i\rangle \leq 0. \qquad (26)$$

On the other hand, since $\tilde{\xi} \in D_\Delta^\epsilon$, we have $\|\tilde{\xi}\| \leq \gamma$ by (8). Then we could choose a sufficiently small $\delta' > 0$ and $\xi \in \Xi$ in (25) such that $\|\xi\| \leq \gamma + \theta$ where $\theta$ is given in the definition of $M_i$ and $\xi$ is given by: $\lambda_i = (1+\delta')\tilde{\lambda}_i$, $\lambda_j = \tilde{\lambda}_j$ for $j \neq i$, $w = \tilde{w}$, $\mu = \tilde{\mu}$. Following the same lines toward (26), it gives that $-\delta\langle \Delta + \tilde{x}_i - \tilde{x}_{i_D}, \tilde{\lambda}_i\rangle \leq 0$. Hence, it holds that $\langle -\Delta - \tilde{x}_i + \tilde{x}_{i_D}, \tilde{\lambda}_i\rangle = 0$. The rest of the proof is analogous and thus omitted. ■

**Claim 2:** $\tilde{x}$ is primal feasible to the problem $(P_\Delta)$.

*Proof:* We have known that $\tilde{x}_i \in X$. We proceed to show $-\Delta - \tilde{x}_i + \tilde{x}_{i_D} \leq 0$ by contradiction. Since $\|\tilde{\xi}\| \leq \gamma$, we could choose a sufficiently small $\delta' > 0$ and $\xi$ with $\|\xi\| \leq \gamma + \theta$ in (25) as follows: if $(-\Delta - \tilde{x}_i + \tilde{x}_{i_D})_\ell > 0$, then $(\lambda_i)_\ell = (\tilde{\lambda}_i)_\ell + \delta'$; otherwise, $(\lambda_i)_\ell = (\tilde{\lambda}_i)_\ell$, and $w = \tilde{w}$, $\mu = \tilde{\mu}$. The rest of the proofs is analogous to Claim 1.

Similarly, one can show $g(\tilde{x}_i) \leq 0$ and $-\Delta + \tilde{x}_i - \tilde{x}_{i_D} \leq 0$ by applying analogous arguments. ■

**Claim 3:** It holds that $p_\Delta^* \leq \sum_{i\in V} f_i(\tilde{x}_i) \leq p_\Delta^* + N\epsilon$.

*Proof:* Since $\tilde{x}$ is primal feasible, then $\sum_{i\in V} f_i(\tilde{x}_i) \geq p_\Delta^*$. On the other hand, $\sum_{i\in V} f_i(\tilde{x}_i) = \sum_{i\in V}\mathcal{L}_i(\tilde{x}_i, \tilde{\xi}_i) \leq \sum_{i\in V} Q_i(\tilde{\xi}_i) + N\epsilon \leq p_\Delta^* + N\epsilon$. ■

## V. An illustrative example

In this section, we examine a numerical example to illustrate the performance of our algorithm. Consider a network of four agents and let the objective functions of agents $f_i : \mathbb{R}_{\geq 0} \to \mathbb{R}$

be equal and defined as follows:

$$f_1(z) = \begin{cases} 0, & z \in [0,1], \\ z-1, & z \in [1,2], \\ 1, & z \in [2,+\infty). \end{cases}$$

$$f_2(z) = \begin{cases} 1, & z \in [0,2], \\ z-1, & z \in [2,3], \\ 2, & z \in [3,+\infty). \end{cases}$$

$$f_3(x) = (z+0.25)^2, \quad f_4(x) = (z-0.5)^2.$$

It is easy to verify that $f_1$ and $f_2$ are not convex and $f_3$ and $f_4$ are convex. The primal problem of interest is given by:

$$\min_{z\in\mathbb{R}} \sum_{i=1}^{4} f_i(z), \quad \text{s.t.} \quad z \in X := [0,10]. \tag{27}$$

The objective function of problem (27) is piecewise convex, and it is not difficult to check that it has a unique solution $z = \frac{1}{8}$ and the optimal value is $p^* = \frac{41}{32} \approx 1.2813$. The associated approximate problem to (27) is then:

$$\begin{aligned} &\min_{x\in\mathbb{R}^4} \sum_{i=1}^{4} f_i(x_i), \\ &\text{s.t.} \quad x_i \in X := [0,10], \quad \forall i \in V, \\ &x_1 - x_2 \le \delta, \quad x_2 - x_1 \le \delta, \\ &x_2 - x_3 \le \delta, \quad x_3 - x_2 \le \delta, \\ &x_3 - x_4 \le \delta, \quad x_4 - x_3 \le \delta, \\ &x_4 - x_1 \le \delta, \quad x_1 - x_4 \le \delta, \end{aligned} \tag{28}$$

where the scalar $\delta = 1$. It can be seen that for any value $\bar{z} \in X$, $\bar{x} = [\bar{z}\ \ \bar{z}\ \ \bar{z}\ \ \bar{z}]^T$ is a Slater vector of problem (28) and all the agents can agree upon the value $\bar{z}$ through the max-consensus algorithm within a finite number of iterations. Here, we choose $\bar{z} = 0.5$. We further choose the tolerance level $\epsilon = 0.1$, and then compute $\gamma_1 = \gamma_2 = \frac{\epsilon}{\delta}$ and $\gamma_3 = \frac{0.75^2+\epsilon}{\delta}$, $\gamma_4 = \frac{\epsilon}{\delta}$. Therefore, we have $\gamma = 4\max_{i\in V}\gamma_i = 4\frac{0.75^2+\epsilon}{\delta} = 2.65$ and then choose $\theta = 0.35$ for the set $M_i$.

We now proceed to check the strong duality of problem (28). To do this, we first define the Lagrangian function $\mathcal{L}$ as follows:

$$\begin{aligned} \mathcal{L}(x,\xi) = \sum_{i\in V} \big(& f_i(x_i) \\ &+ \langle \lambda_1, -x_1 + x_2 - \delta\rangle + \langle w_1, x_1 - x_2 - \delta\rangle \\ &+ \langle \lambda_2, -x_2 + x_3 - \delta\rangle + \langle w_2, x_2 - x_3 - \delta\rangle \\ &+ \langle \lambda_3, -x_3 + x_4 - \delta\rangle + \langle w_3, x_3 - x_4 - \delta\rangle \\ &+ \langle \lambda_4, -x_4 + x_1 - \delta\rangle + \langle w_4, x_4 - x_1 - \delta\rangle\big), \end{aligned}$$

where $\xi := (\lambda_i, w_i)_{i\in V}$. The dual function is given by $Q(\xi) = \inf_{x\in X^4} \mathcal{L}(x,\xi)$. Notice that $Q(0) = \inf_{x\in X^4}\sum_{i\in V} f_i(x_i) = \frac{17}{16}$. The primal and dual optimal values of problem (28) are denoted by $p^*_\delta$ and $d^*_\delta$, respectively. Note that $p^*_\delta = \frac{17}{16} \approx 1.0625$ with $[1\ \ 1\ \ 0\ \ 0.5]^T$ being one of primal solutions and thus $Q(0) = p^*_\delta$. This establishes that $d^*_\delta \ge Q(0) = p^*_\delta$. On the other hand, it follows from weak duality that $p^*_\delta \ge d^*_\delta$. We now conclude that $p^*_\delta = d^*_\delta$ and thus the duality gap of problem (28) is zero.

Figure 1 and 2 show that the evolution of states $x_i(k)$ and the global objective function of problem (28). After 150 iterations, states $x_i(k)$ converge to 0.2436, 0, 0 and 0.1509, respectively which consist of a feasible solution. Figure 2 indicates that $\sum_{i=1}^{4} f_i(x_i(k))$ converges to the value 1.1844 which is in the interval of $[p^*_\delta - 4\epsilon,\ \ p^*_\delta + 4\epsilon] = [0.6625,\ \ 1.4625]$ and is a good approximation of $p^*$ and $p^*_\delta$.

## VI. Conclusion

We have studied a multi-agent optimization problem where the goal of agents is to minimize a sum of local objective functions in the presence of a global inequality constraint and a global state constraint set. Objective and constraint functions as well as constraint sets are not necessarily convex. We have presented the distributed approximate dual subgradient algorithm which allow agents to asymptotically converge to a pair of approximate primal-dual solutions provided that the Slater's condition and strong duality property are satisfied.

## VII. Appendix

### *A. Nonexpansion property of projection operators*

**Lemma 7.1:** [3] Let $Z$ be a non-empty, closed and convex set in $\mathbb{R}^n$. For any $z \in \mathbb{R}^n$, the following holds for any $y \in Z$: $\|P_Z[z] - y\|^2 \le \|z - y\|^2 - \|P_Z[z] - z\|^2$.

### *B. A property of weighted sequence*

**Lemma 7.2:** [29] Consider the sequence $\{\delta(k)\}$ defined by $\delta(k) := \frac{\sum_{\tau=0}^{k-1}\alpha(\tau)\rho(\tau)}{\sum_{\tau=0}^{k-1}\alpha(\tau)}$, where $\rho(k) \in \mathbb{R}^n$, $\alpha(k) > 0$, and $\sum_{k=0}^{+\infty}\alpha(k) = +\infty$. If $\lim\limits_{k\to+\infty}\rho(k) = \rho^*$, then $\lim\limits_{k\to+\infty}\delta(k) = \rho^*$.

### *C. Background on set-valued maps*

We let $\mathbb{X}$ and $\mathbb{Y}$ denote Hausdorff topological spaces. A set-valued map $\Omega : \mathbb{X} \to \mathbb{Y}$ is a map that associates with any $x \in \mathbb{X}$ a subset $\Omega(x)$ of $\mathbb{Y}$. The following definitions and theorem are adopted from [1].

**Definition 7.1:** The set-valued map $\Omega$ is closed at a point $x \in \mathbb{X}$ if $\{x(k)\} \subset \mathbb{X}$, $\lim\limits_{k\to+\infty}\mathrm{dist}(x(k),x) = 0$, $y(k) \in \Omega(x(k))$, and $\lim\limits_{k\to+\infty}\mathrm{dist}(y(k),y) = 0$ implies that $y \in \Omega(x)$.

**Definition 7.2:** The set-valued map $\Omega$ is called upper semicontinuous at $x \in \mathbb{X}$ if and only if any neighborhood $\mathcal{U}$ of $\Omega(x)$, there is $\eta > 0$ such that $\forall x' \in B(x,\eta)$, it holds that $\Omega(x') \subset \mathcal{U}$.

**Theorem 7.1:** Let $\Omega$ and $\Pi$ be two set-valued maps from $\mathbb{X}$ to $\mathbb{Y}$. Assume that $\Omega$ is closed, $\Pi(x)$ is compact and $\Pi$ is upper semicontinuous at $x \in \mathbb{X}$. Then $\Omega\cap\Pi$ is upper semicontinuous at $x$.

### *D. Dynamic average consensus algorithms*

The following is the vector version of the first-order dynamic average consensus algorithm proposed in [30]:

$$x^i(k+1) = \sum_{j=1}^{N} a^i_j(k)x^j(k) + \eta^i(k), \tag{29}$$

where $x^i(k), \eta^i(k) \in \mathbb{R}^n$. Denote $\Delta\eta_\ell(k) := \max_{i\in V} \eta^i_\ell(k) - \min_{i\in V} \eta^i_\ell(k)$ for $1 \leq \ell \leq n$.

**Proposition 7.1:** [30] Let the periodic strong connectivity assumption 2.3, the non-degeneracy assumption 2.1 and the balanced communication assumption 2.2 hold. Assume that $\lim_{k\to+\infty} \Delta\eta_\ell(k) = 0$ for all $1 \leq \ell \leq n$ and all $k \geq 0$. Then the implementation of Algorithm (29) achieves consensus, i.e., $\lim_{k\to+\infty} \|x^i(k) - x^j(k)\| = 0$ for all $i, j \in V$.

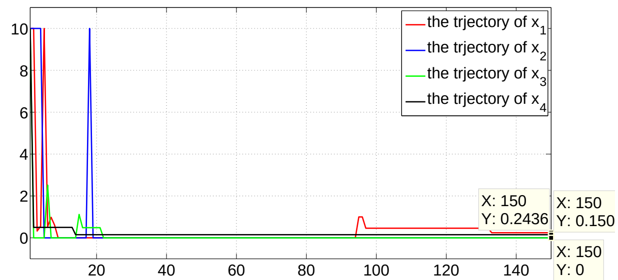


Fig. 1. The evolution of states with the convergent vector of $[0.2436\ \ 0\ \ 0\ \ 0.1509]^T$

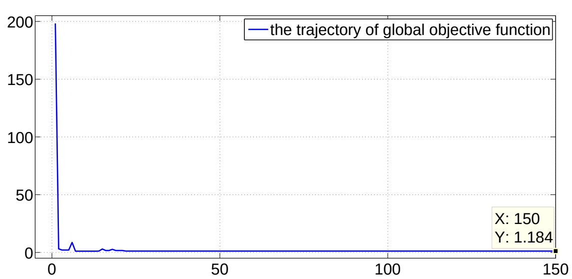


Fig. 2. The evolution of the global objective function along the trajectories of the system with the convergent value of $1.184$.

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